# MATH 2263

## Differential Equations and Dynamical Systems

## Final Exam

January 10, 2018 09:30-11:00

Student Name:________________________________________

Department:________________________________________

Instructors: *Prof. Dr. Erol Sezer, Assist. Prof. Dr. Ahmet Yantir*

| Question# | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Question Value | 25 | 25 | 25 | 25 | 100 |
| Your Grade | | | | | |

INSTRUCTIONS

1. 90 mins, Open book/notes. No cell phones.
2. Answer each question on its page. Explain your results.
3. Mark your instructor.
4. Observe the honor code. Read and sign the statement below.

No help is received, given, or observed.

SIGN:

[25pt] 1. The differential equation

$$x' = 4(\sqrt{x} - \sqrt{x-9} - 1)$$

has a unique equilibrium point $x_e = 25$. Use linearization to find an approximate expression for the solution of the equation corresponding to $x(0) = 24$.

SOLUTION

With $\tilde{x} = x - x_e = x - 25$ 5

$\tilde{x}' \approx a\tilde{x}$, $a = \left[\frac{d}{dx} 4(\sqrt{x} - \sqrt{x-9} - 1)\right]_{x=25} = 4\left[\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-9}}\right]_{x=25} = -0.1$

10

$\tilde{x}(0) = x(0) - 25 = -1$

Thus 10

$\tilde{x} \approx -e^{-0.1t}$, $x \approx 25 - e^{-0.1t}$

[25pt] 2. Solve the system of following differential equations for $y$:

$$\begin{aligned} \dot{x} &= -x + 2y, & x(0) &= 1 \\ \dot{y} &= -5x - 3y + 13, & y(0) &= 1 \end{aligned}$$

<u>SOLUTION</u>

$$sX(s) - 1 = -X(s) + 2Y(s)$$

$$sY(s) - 1 = -5X(s) - 3Y(s) + \frac{13}{s}$$

$$X = \frac{1+2Y}{s+1} \Rightarrow \left(s+3+\frac{10}{s+1}\right)Y = 1 - \frac{5}{s+1} + \frac{13}{s}$$

$$\Rightarrow Y = \frac{s^2+9s+13}{s(s^2+4s+13)} = \frac{1}{s} + \frac{5}{s^2+4s+13}$$

$$= \frac{1}{s} + \frac{5}{(s+2)^2+3^2}$$

$$\Rightarrow y = 1 + \frac{5}{3}e^{-2t}\sin 3t$$

[25pt] 3. The third order homogenous differential equation

$$y^{(3)} + a_1 y'' + a_2 y' + a_3 y = 0$$

has a solution $y = e^{-t}(1+\sin t)$ corresponding to some set of initial conditions $y(0) = y_0$, $y'(0) = y_1$, $y''(0) = y_2$.

(a) Determine $a_1$, $a_2$, $a_3$.

(b) Find the solution corresponding to $y_0 = 1$, $y_1 = y_2 = 0$.

SOLUTION

a) $s^3 + a_1 s^2 + a_2 s + a_3 = (s+1)(s^2+2s+2) = s^3 + 3s^2 + 4s + 2$

b) $s^3 Y - y''(0) - s y'(0) - s^2 y(0) + 3\left[s^2 - y'(0) - s y(0)\right]$
$+ 4\left[sY - y(0)\right] + 5Y = 0$

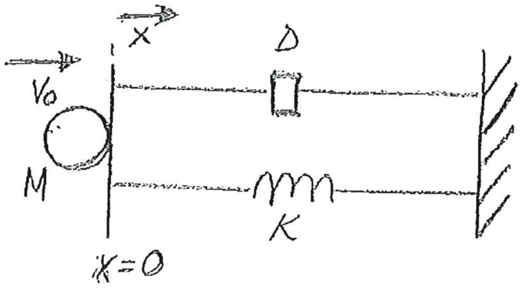
$\Rightarrow (s+1)(s^2+2s+2)Y = s^2 + 3s + 4$

$\Rightarrow Y = \dfrac{s^2+3s+4}{(s+1)(s^2+2s+2)} = \dfrac{2}{s+1} + \dfrac{-s}{s^2+2s+2}$

$= \dfrac{2}{s+1} + \dfrac{-(s+1)+1}{(s+1)^2+1^2}$

$\Rightarrow y = e^{-t}(2 - \cos t + \sin t)$

[25pt] 4. An object moving in the $x$ direction with a velocity of $v_0 = 1$ m/s hits at $t = 0$ to a plate placed at the point $x = 0$ and supported by a spring and shock observer and sticks to the plate as shown in the following figure.




$$M = 1\ kg$$
$$K = 2\ N/m$$
$$D = 3\ kg/s$$
$$v_0 = 1\ m/s.$$

(a) Obtain a differential equation that models the motion of the object for $t \geq 0$.

(b) Find the position and velocity of the object as a function of time.

(c) Calculate the maximum distance the object moves in $x$ direction.

(d) Find the final position and velocity of the object.

*GOOD LUCK!*

SOLUTION

a) $\left.\begin{array}{l} M\ddot{x} + D\dot{x} + Kx = 0 \\ x(0) = 0 \\ \dot{x}(0) = V_0 \end{array}\right\} \Rightarrow \begin{array}{l} \ddot{x} + 3\dot{x} + 2x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 1 \end{array}$

b) $x = c_1 e^{-t} + c_2 e^{-2t}$

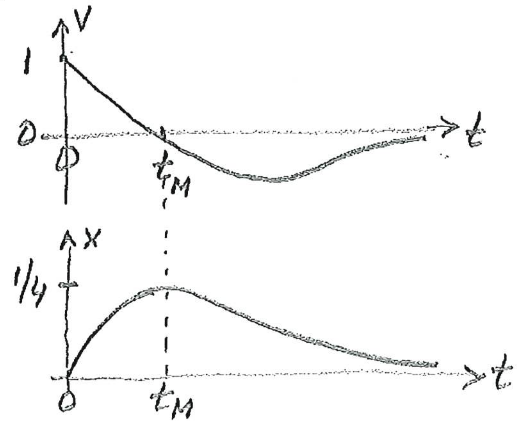
I.C. $\Rightarrow \begin{array}{l} c_1 + c_2 = 0 \\ -c_1 - 2c_2 = 1 \end{array} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array} \Rightarrow \begin{array}{l} x = e^{-t} - e^{-2t} \\ V = -e^{-t} + 2e^{-2t} \end{array}$

c) $\dot{x} = 0 \Rightarrow -e^{-t} + 2e^{-2t} = 0 \Rightarrow e^{-t_M} = \frac{1}{2}$

$x(t_M) = e^{-t_M} - e^{-2t_M} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

d) $x_f = \lim_{t \to \infty} x(t) = 0$

$V_f = \lim_{t \to \infty} V(t) = 0$



# MATH 263

## Differential Equations and Dynamical Systems

## Final Exam

December 27, 2017 12:30-14:00

Student Name:________________________________________

Department:__________________________________________

**Instructors:** *Prof. Dr. Erol Sezer, Assist. Prof. Dr. Ahmet Yantir*

| Question# | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Question Value | 25 | 25 | 25 | 25 | 100 |
| Your Grade | | | | | |

INSTRUCTIONS

1. 90 mins, Open book/notes. No cell phones.
2. Answer each question on its page. Explain your results.
3. Mark your instructor.
4. Observe the honor code. Read and sign the statement below.

   No help is received, given, or observed.

   SIGN:

[25pt] 1. Consider the nonlinear system

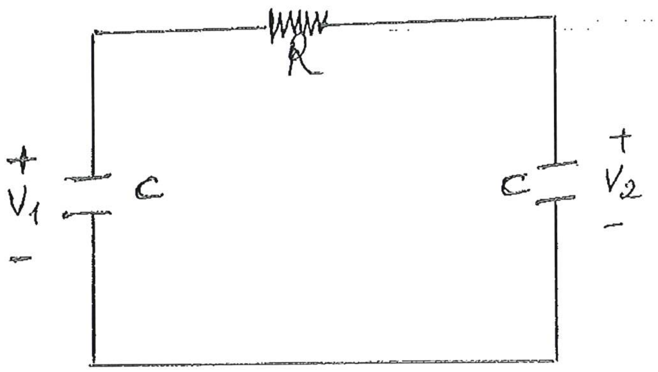
$$\begin{aligned} x' &= 2y - 6x \\ y' &= 4 - x^2 \end{aligned}$$

(a) Find and classify the equilibrium points.
(b) Find an approximate solution with the initial conditions $x(0) = 2.1,\ y(0) = 6.1$.

[25pt] 2. Find $x(0.1)$ and $y(0.1)$ using modified (generalized) Euler method with stepsize $h = 0.1$.

$$x' = 4 - y, \quad x(0) = 0$$
$$y' = 2 - x, \quad y(0) = 0.$$

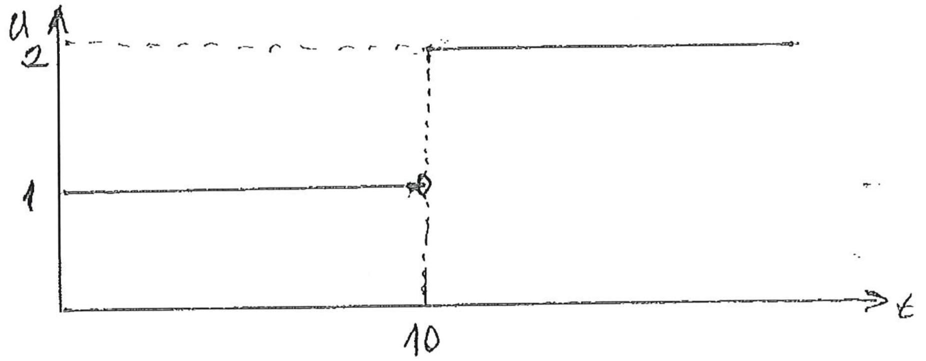
[25pt] 3. In the following circuit $R = 1\Omega$, $C = 1F$, $v_1(0) = 10V$ and $v_2(0) = 2V$. Find $v_1(t)$ and $v_2(t)$ nd plot them in the same graph.



[25pt] 4. Consider the following differential equation

$$x' + x = u, \quad x(0) = 0$$

where u is given by the following graph:




Find and roughly plot $\ddot{x}(t)$.

*GOOD LUCK!*

EEE 205

DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

FINAL EXAM

13 January 2016, Dr. M. Erol Sezer

INSTRUCTIONS:

1. 90 min, Open book/notes, No cell phones/calculators.
2. Answer each question on its page. Explain your results.
3. Observe the honor code. Read and sign the statement below.

   No help is received, given, or observed.

   Signed:

| NAME | | | | | | |
|---|---|---|---|---|---|---|
| QUESTION | 1 | 2 | 3 | 4 | 5 | TOTAL |
| OUT OF | 5 | 5 | 5 | 5 | 10 | 30 |
| GRADE | | | | | | |

1. Solve

$$y' = -y + 2\cos t\,,\quad y(0) = 1$$

Solution :

$$sY(s) - 1 = -Y(s) + \frac{2s}{s^2+1}$$

$$(s+1)\,Y(s) = 1 + \frac{2s}{s^2+1} = \frac{(s+1)^2}{s^2+1}$$

$$Y(s) = \frac{s+1}{s^2+1}$$

$$y(t) = \cos t + \sin t$$

2. Solve for $x(t)$

$$\begin{aligned} x' &= -2x + y \\ y' &= x - 2y + 6 \end{aligned}$$

with $x(0) = y(0) = 0$.

Solution :

$$\left.\begin{aligned} (s+2)X(s) &= Y(s) \\ (s+2)Y(s) &= X(s) + \frac{6}{s} \end{aligned}\right\} \Rightarrow (s+2)^2 X(s) = X(s) + \frac{6}{s}$$

$$X(s) = \frac{6}{s(s^2+4s+3)} = \frac{6}{s(s+1)(s+3)} = \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3}$$

$$x(t) = 2 - 3e^{-t} + e^{-3t}$$

3. Plot phase portraits of the system

$$x' = y - 1$$
$$y' = 1 - x$$

Solution :

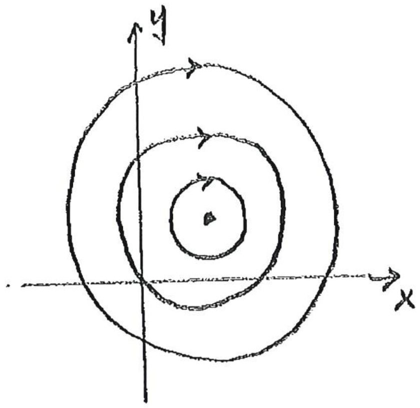
With $u = x-1$, $v = y-1$,

$$\left.\begin{aligned} u' &= v \\ v' &= -u \end{aligned}\right\} \Rightarrow u^2 + v^2 = c^2$$

$$\Rightarrow (x-1)^2 + (y-1)^2 = c^2$$

Circles centered at $x_e = 1$, $y_e = 1$

Directions: By inspection



4. Find equilibria of the system and analyze stability properties

$$x' = -2x + \sin x$$

Solution :

$f(x) = -2x + \sin x = 0 \Rightarrow x = x_e = 0$

$f'(x) = -2 + \cos x \Rightarrow f'(x_e) = -1$

Linearized system :

$\tilde{x}' = -\tilde{x} \Rightarrow$ Stable

5. A diesel engine drives a load and an inertia $J = 3\ N.m.s^2$ connected to its shaft. Torque demand of the load is $T_L$. Torque produced by the engine is a function of its angular speed $\omega$, and is given by

$$T(\omega) = 3 \times 10^{-3}\,\omega\,(600 - \omega)$$

(a) Obtain a dynamical model of the system.

(b) Find the equilibrium value $\omega_e$ of $\omega$ for a constant load torque $T_{Le} = 240\ N.m$.

(c) Obtain an approximate expression for $\omega(t)$ if the load torque suddenly drops to $T_L = 210\ N.m$ at $t = 0$.

Solution :

a) $T = J\omega' + T_L$

b) $T(\omega_e) = T_{Le} \Rightarrow 3\times10^{-3}\,\omega_e\,(600-\omega_e) = 240$

$\Rightarrow \omega_e = 400$

c) With $\omega = \omega_e + \tilde{\omega}$, $T_L = T_{Le} + \tilde{T}_L$

$J\tilde{\omega}' + T_{Le} + \tilde{T}_L = T(\omega_e + \tilde{\omega}) \cong T(\omega_e) + T'(\omega_e)\tilde{\omega}$

$J\tilde{\omega}' = T'(\omega_e)\tilde{\omega} - \tilde{T}_L$

$T'(\omega) = 1.8 - 6\times10^{-3}\omega \Rightarrow T'(\omega_e) = -0.6$

$3\tilde{\omega}' = -0.6\tilde{\omega} + 30$

$\Rightarrow \tilde{\omega}' = -0.2\tilde{\omega} + 10$, $\tilde{\omega}(0) = 0$

$\Rightarrow \tilde{\omega} = 50\,(1 - e^{-0.2t})$

$\Rightarrow \omega(t) \cong \omega_e + \tilde{\omega} = 450 - 50\,e^{-0.2t}$

MATH 263

# Differential Equations and Dynamical Systems

## Final Exam

**05 January 2017 14:30 - 16:00**

**Student Name:**________________________________________

**Instructors:** *Prof. Dr. Erol Sezer, Assist. Prof. Dr. Ahmet Yantir*

| Question# | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Question Value | 25 | 25 | 25 | 25 | 100 |
| Your Grade | | | | | |

INSTRUCTIONS

1. 90 mins, Open book/notes. No cell phones/calculators
2. Answer each question on its page. Explain your results.
3. Observe the honor code. Read and sign the statement below.

**No help is received, given, or observed.**

**SIGN:**

[25pt] 1. Consider the linear nonhomogeneous system

$$\frac{dx}{dt} = x + y - 2$$
$$\frac{dy}{dt} = x - y.$$

(a) Determine the equilibrium point $\mathbf{x}^0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

(b) Classify its type and examine its stability by making transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}.$$

(c) Plot roughly its phase plane.

a) $\left.\begin{array}{l} x+y-2=0 \\ x-y=0 \end{array}\right\} \Rightarrow 2x=2 \Rightarrow x=1 \Rightarrow (1,1)$ is the equilibrium pt. of the system.

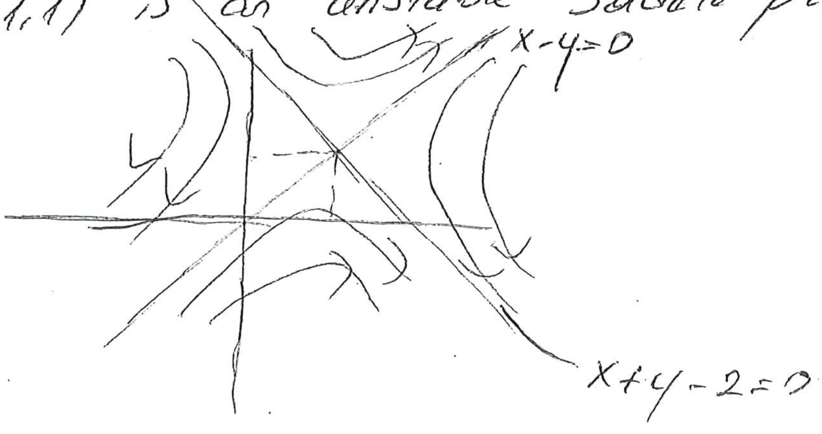
b) $\begin{array}{l} x = u+1 \\ y = v+1 \end{array} \Rightarrow$ the system reduces to

$$\frac{du}{dt} = u+v$$

$$\frac{dv}{dt} = u-v$$

$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ has eigenvalues $\lambda_1 = \sqrt{2} > 0$, $\lambda_2 = -\sqrt{2} < 0$

$\Rightarrow (1,1)$ is an unstable saddle pt.



[25pt] 2. Consider the linear nonhomogeneous system

$$\frac{dx}{dt} = -y + e^{-t}$$
$$\frac{dy}{dt} = x + 3e^{-t}.$$

(a) Write this system as a second order linear equation.

(b) Solve the system.

a) $y'' = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(x + 3e^{-t}\right) = \frac{dx}{dt} - 3e^{-t}$

$= -y + e^{-t} - 3e^{-t}$

$\Rightarrow y'' + y = -2e^{-t}$

b) By method of undetermined coefficients

$y(t) = c_1 \cos t + c_2 \sin t - e^{-t}$ //

$\Rightarrow \frac{dx}{dt} = -c_1 \cos t - c_2 \sin t + e^{-t} + e^{-t}$

$\Rightarrow x(t) = -c_1 \sin t + c_2 \cos t - 2e^{-t}$

[25pt] 3. Find the equilibrium points of the nonlinear system

$$\frac{dx}{dt} = x + y^2$$
$$\frac{dy}{dt} = x + y.$$

Analyze their stability by linearizing system.

$\left.\begin{array}{l} x+y^2=0 \\ x+y=0 \end{array}\right\} \Rightarrow y^2-y=0 \Rightarrow y(y-1)=0 \Rightarrow y=0$ or $y=1$.

$\Rightarrow (0,0)$ and $(-1,1)$ are equilibrium pts.

For $(0,0)$

$$A = J\Big|_{(0,0)} = \begin{pmatrix} F_x\big|_{(0,0)} & F_y\big|_{(0,0)} \\ G_x\big|_{(0,0)} & G_y\big|_{(0,0)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Eigen values of $A$ are $\lambda_1 = \lambda_2 = 1 > 0 \Rightarrow (0,0)$ is an unstable equilibrium pt.

For $(-1,1)$

$$A = J\Big|_{(-1,1)} = \begin{pmatrix} F_x\big|_{(-1,1)} & F_y\big|_{(-1,1)} \\ G_x\big|_{(-1,1)} & G_y\big|_{(-1,1)} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Eigenvalues of $A$ are $\left.\begin{array}{l} \lambda_1 = 1+\sqrt{2} > 0 \\ \lambda_2 = 1-\sqrt{2} < 0 \end{array}\right\} \Rightarrow (-1,1)$ is an unstable equilibrium pt.

[25pt] 4. Consider the following second order, variable coefficient, linear homogenous equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0.$$

(a) Using the transformation $x = e^t$ or $t = \ln x$, reduce the above equation to a second order, constant coefficient, linear homogenous equation.

(b) Find its general solution.

a) Let $x = e^t \Rightarrow t = \ln x$. By chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

Product Rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\cdot\frac{dy}{dt} + \frac{d^2y}{dt^2}\cdot\frac{dt}{dx}\cdot\frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

Hence the given D.E reduces to

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt}\right) - 3x\left(\frac{1}{x}\frac{dy}{dt}\right) + 3y = 0$$

which gives us

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0.$$

b) Characteristic eqn: $r^2 - 4r + 3 = 0 \Rightarrow r_1 = 1$ and $r_2 = 3$

$$\Rightarrow y(t) = c_1 e^t + c_2 e^{3t}$$

$$\Rightarrow y(x) = c_1 x + c_2 x^3 \;//$$

GOOD LUCK!